Identify flow structures with Proper Orthogonal Decomposition (POD)

Knud Erik Meyer
Department of Mechanical Engineering, Technical University of Denmark (DTU)

Coworkers: Jakob M. Pedersen, Oktay Özcan, Sajjad Haider and Poul S. Larsen
Information in snapshots

World of RANS and LDA

World of LES and PIV

Mean velocity field

Instantaneous velocities (snapshot)

\( V/U_\infty \)

Black line indicates jet trajectory
Proper Orthogonal Decomposition (snapshot)

Arrange fluctuating part of velocity components for the $N$ snapshots as:

$$U = [u^1 \ u^2 \ \ldots \ u^N] = \begin{bmatrix}
    u_1^1 & u_1^2 & \ldots & u_1^N \\
    \vdots & \vdots & \ddots & \vdots \\
    u_l^1 & u_l^2 & \ldots & u_l^N \\
    v_1^1 & v_1^2 & \ldots & v_1^N \\
    \vdots & \vdots & \ddots & \vdots \\
    v_l^1 & v_l^2 & \ldots & v_l^N
\end{bmatrix}$$

Calculate the matrix:

$$\tilde{C} = U^T U$$

Solve the eigenvalue problem:

$$\tilde{C} A^i = \lambda^i A^i$$

Arrange solutions by eigenvalue as:

$$\lambda_1 > \lambda_2 > \ldots > \lambda_N = 0$$

POD modes are now found as:

$$\phi^i = \frac{\sum_{n=1}^{N} A_n^i u^n}{\left|\sum_{n=1}^{N} A_n^i u^n\right|}, \quad i = 1, \ldots, N$$
POD coefficients and reconstruction

With POD modes arranged as

$$\Psi = \begin{bmatrix} \phi^1 & \phi^2 & \ldots & \phi^N \end{bmatrix}$$

POD coefficients $a_i$ can be found for the snapshot $n$ as

$$a_i = \phi^i u^n \quad \text{or} \quad a^n = \Psi^T u^n$$

A snapshot (fluctuating part) can be reconstructed as

$$u^n = \sum_{i=1}^{N} a_i^n \phi^i = \Psi a^n$$
MATLAB script example

% create matrix will all fluctuating velocity components for each snapshot in a column
Uall=[reshape(Uf,ni*nj,ns);reshape(Vf,ni*nj,ns)];

% Do POD analysis
R=Uall'*Uall;         % Autocovariance matrix
[eV,D]=eig(R);        % solve: eV is eigenvectors, D is eigenvalues in diagonal matrix
[L,I]=sort(diag(D));  % sort eigenvalues in ascending order - I is sorted index vector
for i=1:length(D)
    eValue(length(D)+1-i)=L(i);      % Eigenvalues sorted in descending order
    eVec(:,length(D)+1-i)=eV(:,I(i)); % Eigenvectors sorted in the same order
end;
eValue(length(eValue))=0;   % last eigenvalue should be zero
energy=eValue/sum(eValue); % relative energy associated with mode m

% calculate the first 10 modes
for i=1:10
    tmp=Uall*eVec(:,i);         % find mode
    phi(:,i)=tmp/norm(tmp);     % normalize mode
end;

only 14 lines of code → programming POD is not very difficult!
POD properties

- Eigenvalue reflects relative kinetic energy associated with corresponding mode

- A snapshot may often be reconstructed satisfactorily using a few of the most energetic modes

- The most energetic modes can be viewed as "dominant" flow structures
Cylindrical cavity with rotation lid

• "Vortex in a cylinder"

• Looking for instabilities in laminar flow

• Well-know vortex breakdown, but is there non-axisymmetric instabilities?

• LDA finds frequencies, but is this in oscillations in time or a rotating structure?
Mean field and snapshot
Fluctuating part

snapshot - fluctuation from mean field
First two POD modes
Energy distribution on modes

Relative energy of POD modes

% energy

mode no.
Coefficients variation in time

Time variation of the POD coefficients

- $a_1$
- $a_2$
Swirling flow through cylinder

- Model of cavenging of two-stroke Diesel engine
- Turbulent flow with helical vortex
- Need to characterize vortex movement.
Similar to bathtub vortex?
Mean flow

Mean field

color show vorticity
Snapshot

color show vorticity
POD modes

color show vorticity
Reconstruction of snapshot
Data analysis using POD

- Detect vortex position on snapshots and track vortex in time

- Evaluate fluctuating energy associated with vortex – and the turbulence kinetic energy excluding the vortex movements

- Other modes suggest dual vortices and other phenomena
Flow structures in a jet in crossflow

- jet shear layer vortices
- horseshoe vortex
- hanging vortex
- wake vortices
- counter rotating vortex pair (CVP)
Visualization in turbulent flows

- Traditional visualization experiments use *laminar* in-flow

- Real applications have *turbulent* in-flow

- Studies and comparison needs new tool – we will try Proper Orthogonal Decomposition (POD).
Stereoscopic PIV in wind tunnel

Re_D = 2400
D = 24 mm
U_m = 5 m/s

U_∞ = 1.5 m/s
Turbulent boundary layer

light sheet optics

camera 1

camera 2
Traditional comparison – mean velocity

Solid lines are LES and circles are LDA data – from the y=0 plane
Input data for POD

Mean velocity field

Instantaneous velocities (snapshot)

line indicates jet trajectory
POD-modes at $y=0$ plane

Mode 1: 11.5%

Mode 2: 9.8%

Mode 3: 3.4%

Mode 5: 3.0%
POD at the $z=1.33D$ plane
POD-modes at x=D plane
The two first POD coefficients from the snapshots from the $z=1.33D$ plane
Snapshot $z=1.33D$ (1/4)

Snapshot:

Reconstruction:

\[(a_1, a_2) = (12.0, -0.1)\]
Snapshot $z=1.33D$ (2/4)

snapshot:

reconstruction:

$(a_1,a_2)=(10.1,8.2)$
Snapshot $z=1.33D$ (3/4)

snapshot:

reconstruction:

$(a_1,a_2)=(0.4,11.0)$
Snapshot $z=1.33D$ (4/4)

Snapshot:

Reconstruction:

$(a_1, a_2) = (-9.5, 8.4)$
Animation based on two modes
PIV-LES comparison – y=0 plane

mode 1 with 11.5% of energy

mode 1 with 8.4% of energy

mode 4 with 3.1% of energy

mode 4 with 2.9% of energy
$a_1$ and $a_2$ coefficients from LES

![Graph showing $A_1$ and $A_2$ coefficients over snapshot number]
Conclusions – jet in crossflow

• POD on different planes agree on first two modes

• Semi-periodic flow features can be guessed from non-time resolving measurements

• Most important modes are connected to generation of wake vortices – involves jet core.

• Despite some difference in mean field, POD shows that PIV dynamics agrees well with LES

LES calculations

- Finite volume Navier-Stokes solver
- Mixed Scale Eddy Viscosity model
- 4.7 million grid cells
- Special runs to generate inlet conditions
- POD data based on run corresponding to 90 statistically independent samples.
PIV measurements

- 200 mJ Nd:YAG laser
- Two Kodak MegaPlus ES 1.0 cameras (1 MPix)
- Calibration target

- All three velocity components in planes
- Planes in all three main directions

- 1000 samples